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Mixed boundary-value problems in the theory of elasticity of thin laminated bodies of variable thickness consisting of anisotropic inhomogeneous materials☆

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ABSTRACT

Starting from the three-dimensional equations of the theory of thermoelasticity, two-dimensional equations for thin laminated bodies are derived in a general formulation and solved by an asymptotic method. The bodies and layers, consisting of anisotropic and inhomogeneous materials (with respect to two longitudinal coordinates), bounded by arbitrary smooth non-intersecting surfaces, also have variable thicknesses. Recursion formulae are derived for determining the components of the stress tensor and the displacement vector when the kinematic or mixed boundary conditions of the static boundary-value problem of the theory of thermoelasticity are specified on the faces of the body, assuming that the corresponding heat conduction problem is solved. An algorithm for constructing of the analytical solutions of the boundary-value problems formulated is developed using modern computational facilities.
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The asymptotic method of solving boundary-value problems in the theory of thermoelasticity for anisotropic strips, plates and shells has turned out to be very effective for solving both static[1–7] and dynamic[8–10] problems. In several cases, when the functions specified on the faces are polynomials, the iterative process is terminated after a finite number of steps and leads to a mathematically exact solution for a strip and a layer[7]. It seems worth using the proposed asymptotic method and the possibilities of modern computational techniques to derive a universal algorithm suitable for determining the stress-strain state of an arbitrary thin body consisting of anisotropic inhomogeneous layers.

1. Formulation of the boundary-value problems

A thin body, consisting of n layers bounded by smooth non-intersecting surfaces, is considered in the system of coordinates $OXYZ$

$$\varphi_0(x,y)<\varphi_1(x,y)<\varphi_2(x,y)<\cdots<\varphi_{n-1}(x,y)<\varphi_n(x,y)$$
$$|x|\le a,\quad |y|\le b\quad (\text{or } |x|<\infty, |y|<\infty) \tag{1.1}$$

The layers consist of dissimilar materials which are anisotropic (21 coefficients of elasticity) and inhomogeneous with respect to the x and y coordinates. It is assumed that the coefficients of elasticity of the layer materials do not differ in order of magnitude.

Specified bulk forces are applied to the layer with number i

$$P_j^{(i)}(x,y,z),\quad j=x,y,z,\quad i=1,2,\ldots,n$$

as well as thermal actions, the effect of which is taken into account using the theory of thermal stresses in accordance with the Duhamel–Neumann law[11] assuming that the temperature function satisfies the heat conduction equation and the contact conditions for layers made of anisotropic and inhomogeneous materials.

Suppose the components of the displacement vector are given on one face of the body (the surface layer with $i=1$) $z=\varphi_0\,(x, y)$ and, in particular, this surface can be rigidly clamped (the displacement vector is equal to zero):

$$z=\varphi_0:\quad u_j=u_j^-(x,y),\quad j=x,y,z \tag{1.2}$$

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and the static conditions of the first boundary-value problem of the theory of elasticity are specified on the opposite face $z=\varphi_n\,(x,y)$

$$z=\varphi_n:\sigma_{jx}\cos(\vartheta_n,x)+\sigma_{jy}\cos(\vartheta_n,y)+\sigma_{jz}\cos(\vartheta_n,z)=\Phi_{\vartheta j}(x,y)$$
$$j=x,y,z \tag{1.3}$$

the kinematic conditions of the second boundary-value problem

$$z=\varphi_n:u_j=\overset{+}{u}_j(x,y),\quad j=x,y,z \tag{1.4}$$

or one of the combinations ((1.5) or (1.6)) of the mixed conditions

$$z=\varphi_n:u_j=\overset{+}{u}_j(x,y),\quad j=x,y$$
$$\sigma_{xz}\cos(\vartheta_n,x)+\sigma_{yz}\cos(\vartheta_n,y)+\sigma_{zz}\cos(\vartheta_n,z)=\Phi_{\vartheta z}(x,y) \tag{1.5}$$

$$z=\varphi_n:u_z=\overset{+}{u}_z(x,y)$$
$$\sigma_{jx}\cos(\vartheta_n,x)+\sigma_{jy}\cos(\vartheta_n,y)+\sigma_{jz}\cos(\vartheta_n,z)=\Phi_{\vartheta j}(x,y),\quad j=x,y \tag{1.6}$$

$$\cos(\vartheta_i,x)=-\frac{1}{\lambda_i}\frac{\partial\varphi_i}{\partial x}=\frac{1}{\lambda_i}\psi_{ix}(x,y),\quad \cos(\vartheta_i,z)=\frac{1}{\lambda_i}$$
$$\lambda_i=\sqrt{1+\left(\frac{\partial\varphi_i}{\partial x}\right)^2+\left(\frac{\partial\varphi_i}{\partial y}\right)^2},\quad i=1,2,\ldots,n \tag{1.7}$$

$\Phi_{\vartheta j}(x,y)$, $u_j^{\pm}(x,y)$ are given sufficiently smooth functions.

It is required to determine the stress-strain state of a body if the conditions of total contact between the layers

$$z=\varphi_i:\left(\sigma_{jx}^{(i)}-\sigma_{jx}^{(i+1)}\right)\psi_{ix}+\left(\sigma_{jy}^{(i)}-\sigma_{jy}^{(i+1)}\right)\psi_{iy}+\sigma_{jz}^{(i)}-\sigma_{jx}^{(i+1)}=0$$
$$u_j^{(i)}=u_j^{(i+1)},\quad j=x,y,z,\quad i=1,2,\ldots,n-1 \tag{1.8}$$

are satisfied.

The solution must also satisfy the conditions on the ends of the body. These conditions are not presented here since it has been proved[12,13] that they have no effect on the inner solution of the boundary-value problems which have been formulated and the appearance of the boundary layer caused by them, the values of the quantities in which decrease exponentially along the direction of the inward normal to the end plane.

2. Derivation of the resolving equations and the general integral

In the equilibrium equations when account is taken of bulk forces

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+P_x=0\quad(x,y,z) \tag{2.1}$$

and in the equations of state (a generalized Hooke's law) when account is taken of thermal strains[11]

$$\frac{\partial u_x}{\partial x}=e_1+\alpha_{11}\theta,\quad \frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}=e_4+\alpha_{23}\theta\qquad(x,y,z;1,2,3;4,5,6;\alpha_{23},\alpha_{13},\alpha_{12})$$
$$e_m=a_{1m}\sigma_{xx}+a_{2m}\sigma_{yy}+a_{3m}\sigma_{zz}+a_{4m}\sigma_{yz}+a_{5m}\sigma_{xz}+a_{6m}\sigma_{xy},\quad m=1,2,\ldots,6 \tag{2.2}$$

where it is assumed that $a_{ij}(x,y)=a_{ji}(x,y)$, $\alpha_{ij}(x,y)=\alpha_{ji}(x,y)$, $\theta=T-T_0$ is a temperature increment, we will change to the system of dimensionless coordinates ξ, η and ζ and dimensionless variables using the formulae

$$x=l\xi,\quad y=l\eta,\quad z=h\zeta=l\varepsilon\zeta,\quad \varepsilon=h/l,\quad u_x=lu,\quad u_y=l\upsilon,\quad u_z=lw$$
$$h=\operatorname{Sup}|\varphi_i-\varphi_{i-1}|,\quad h\ll l=\min\{a,b\},\quad i=1,2,\ldots,n \tag{2.3}$$

where ε is a small geometric parameter indicating the relative thinness of the laminated body considered.

For an arbitrary i-th layer, we obtain

$$\begin{aligned}
&\frac{\partial\sigma_{jx}^{(i)}}{\partial\xi}+\frac{\partial\sigma_{jy}^{(i)}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{jz}^{(i)}}{\partial\zeta}+lP_j^{(i)}=0,\quad j=x,y,z\\
&\frac{\partial u^{(i)}}{\partial\xi}=e_1^{(i)}+\alpha_{11}^{(i)}\theta^{(i)}\quad(\xi,\eta;u,\upsilon;1,2),\quad \varepsilon^{-1}\frac{\partial w^{(i)}}{\partial\zeta}=e_3^{(i)}+\alpha_{33}^{(i)}\theta^{(i)}\\
&\varepsilon^{-1}\frac{\partial\upsilon^{(i)}}{\partial\zeta}+\frac{\partial w^{(i)}}{\partial\eta}=e_4^{(i)}+\alpha_{23}^{(i)}\theta^{(i)}\quad(\xi,\eta;u,\upsilon;4,5;\alpha_{23},\alpha_{13})\\
&\frac{\partial u^{(i)}}{\partial\eta}+\frac{\partial\upsilon^{(i)}}{\partial\xi}=e_6^{(i)}+\alpha_{12}^{(i)}\theta^{(i)}
\end{aligned}\tag{2.4}$$

The system of equations and relations (2.4) is singularly perturbed by the small geometric parameter ε. Consequently,[4–7,12–15] the solution consists of two solutions: an inner (basic) solution, which predominates within the region occupied by the body, and the solution of the problem for the boundary layer, which decreases exponentially in the direction of the inward normal to the end faces.[4,7] The inner solution is constructed here. The solution for the boundary layer can be found by a previously described method.[4,7]

The solution of the inner problem is sought in the form of an asymptotic expansion[1,4,7]

$$Q^{(i)}=\sum_{s=0}^{S}\varepsilon^{\chi_Q+s}Q^{(i,s)}(\xi,\eta,\zeta)\quad i=1,2,\ldots,n;\quad \chi_u=0,\quad \chi_\sigma=-1\tag{2.5}$$

where $Q^{(i)}$ is any of the stresses and the dimensionless displacements.

The bulk forces and the temperature function are simultaneously represented in the form

$$P_x^{(i)}=\sum_{s=0}^{S}\varepsilon^{-2+s}l^{-1}P_{xs}^{(i)}(\xi,\eta,\zeta)\quad(x,y,z),\quad \theta^{(i)}=\sum_{s=0}^{S}\varepsilon^{-1+s}\theta_s^{(i)}(\xi,\eta,\zeta)\tag{2.6}$$

It follows from representations (2.6) that the order of magnitude of the quantities corresponding to the stress-strain state, caused by the bulk forces and the temperature field, will be of the order of magnitude of the quantities corresponding to the stress-strain state created by the strains and stresses applied to the faces if the intensity of the bulk forces is ε^{-2} times greater and the intensity of the change in the temperature field is ε^{-1} times greater than the order of the dimensionless limiting displacements. Otherwise, the contribution from the first of these will be smaller and the corresponding terms will appear in the equations for the higher-order approximations. Note also that asymptotics (2.5) are true if the coefficients of elasticity of the layers do not differ in their orders of magnitude. It is simple to determine the corresponding asymptotics for exceptional cases.

Substituting expressions (2.5) and (2.6) into system (2.4) and equating coefficients of like powers of ε on both sides of each equation, we obtain a consistent system of equations in the unknown coefficients $Q^{(i,s)}$ of expansion (2.5). Solving this system, we arrive at the following recursion formulae for determining $Q^{(i,s)}$, that is, the components of the stress tensor and the displacement vector

$$\begin{aligned}
&\sigma_{jx}^{(i,s)}=\sigma_{jx0}^{(i,s)}(\xi,\eta)+\sigma_{jx*}^{(i,s)}(\xi,\eta,\zeta),\quad j=x,y,z\\
&\sigma_{xx}^{(i,s)}=\tilde\tau_1^{(i,s)}+\sigma_{xx*}^{(i,s)}(\xi,\eta,\zeta)\quad(xx,yy,xy;1,2,6)\\
&u^{(i,s)}=\zeta\tilde\tau_5^{(i,s)}+u_0^{(i,s)}(\xi,\eta)+u_*^{(i,s)}(\xi,\eta,\zeta)\qquad(u,\upsilon,w;5,4,3)\\
&\tilde\tau_m^{(i,s)}=A_{m3}^{(i)}\sigma_{zz0}^{(i,s)}+A_{m4}^{(i)}\sigma_{yz0}^{(i,s)}+A_{m5}^{(i)}\sigma_{xz0}^{(i,s)},\quad m=1,2,\ldots,6
\end{aligned}\tag{2.7}$$

Here,

$$\begin{aligned}
&\sigma_{jz*}^{(i,s)}=-\int_0^\zeta\left(\frac{\partial\sigma_{jx}^{(i,s-1)}}{\partial\xi}+\frac{\partial\sigma_{jy}^{(i,s-1)}}{\partial\eta}+P_{js}^{(i)}\right)d\zeta,\quad j=x,y,z\\
&\sigma_{xx*}^{(i,s)}=B_{11}^{(i)}R_1^{(i,s)}+B_{12}^{(i)}R_2^{(i,s)}+B_{16}^{(i)}R_3^{(i,s)}\quad(x,y;1,2)\\
&\sigma_{xy*}^{(i,s)}=B_{16}^{(i)}R_1^{(i,s)}+B_{26}^{(i)}R_2^{(i,s)}+B_{66}^{(i)}R_3^{(i,s)}\\
&u_*^{(i,s)}=\int_0^\zeta\left(e_{5*}^{(i,s)}-\frac{\partial w^{(i,s-1)}}{\partial\xi}+\alpha_{13}^{(i)}\theta_s^{(i)}\right)d\zeta\quad(\xi,\eta;u,\upsilon;1,2;5,4)
\end{aligned}\tag{2.8}$$

$$w_*^{(i,s)} = \int_0^\zeta \left(e_{3*}^{(i,s)} + \alpha_{33}^{(i)}\theta_s^{(i)}\right)d\zeta$$

$$e_{m*}^{(i,s)} = a_{m1}^{(i)}\sigma_{xx*}^{(i,s)} + a_{m2}^{(i)}\sigma_{yy*}^{(i,s)} + a_{m3}^{(i)}\sigma_{zz*}^{(i,s)} + a_{m4}^{(i)}\sigma_{yz*}^{(i,s)} + a_{m5}^{(i)}\sigma_{xz*}^{(i,s)} + a_{m6}^{(i)}\sigma_{xy*}^{(i,s)}, \quad m = 3,4,5$$

$$R_1^{(i,s)} = \frac{\partial u^{(i,s-1)}}{\partial\xi} - a_{13}^{(i)}\sigma_{zz*}^{(i,s)} - a_{14}^{(i)}\sigma_{yz*}^{(i,s)} - a_{15}^{(i)}\sigma_{xz*}^{(i,s)} - \alpha_{11}^{(i)}\theta_s^{(i)} \quad (1,2;u,\upsilon;\xi,\eta)$$

$$R_3^{(i,s)} = \frac{\partial u^{(i,s-1)}}{\partial\eta} + \frac{\partial \upsilon^{(i,s)}}{\partial\xi} - a_{36}^{(i)}\sigma_{zz*}^{(i,s)} - a_{46}^{(i)}\sigma_{yz*}^{(i,s)} - a_{56}^{(i)}\sigma_{xz*}^{(i,s)} - \alpha_{12}^{(i)}\theta_s^{(i)}$$

$$B_{pj}^{(i)} = \left(a_{pk}^{(i)}a_{jk}^{(i)} - a_{pj}^{(i)}a_{kk}^{(i)}\right)\Big/\Delta^{(i)} \quad (p \ne j \ne k \ne p)$$
$$B_{kk}^{(i)} = \left(a_{pp}^{(i)}a_{jj}^{(i)} - a_{pj}^{(i)2}\right)\Big/\Delta^{(i)}, \quad B_{pj}^{(i)} = B_{jp}^{(i)}; \quad p,j,k = 1,2,6$$
$$A_{kl}^{(i)} = -a_{1l}^{(i)}B_{k1}^{(i)} - a_{2l}^{(i)2}B_{k2}^{(i)} - a_{6l}^{(i)}B_{k6}^{(i)}$$
$$A_{ml}^{(i)} = a_{ml}^{(i)}A_{1l}^{(i)} + a_{m2}^{(i)}A_{2l}^{(i)} + a_{m6}^{(i)}A_{6l}^{(i)} + a_{ml}^{(i)}, \quad A_{ml}^{(i)} \ne A_{lm}^{(i)}; \quad l,m = 3,4,5, \quad i = 1,2,\ldots,n$$
$$\Delta^{(i)} = a_{11}^{(i)}a_{22}^{(i)}a_{66}^{(i)} + 2a_{12}^{(i)}a_{16}^{(i)}a_{26}^{(i)} - a_{22}^{(i)}a_{16}^{(i)2} - a_{66}^{(i)}a_{12}^{(i)2} - a_{11}^{(i)}a_{26}^{(i)2}$$

The general integral of system of equations (2.1), (2.2) and (2.4) obtained contains 6n as yet unknown integration functions

$$\sigma_{jz0}^{(i,s)}(\xi,\eta), \quad j = x,y,z, \quad u_0^{(i,s)}(\xi,\eta), \quad \upsilon_0^{(i,s)}(\xi,\eta), w_0^{(i,s)}(\xi,\eta); \quad i = 1,2,\ldots,n$$

which are uniquely determined from the contact conditions for the layers (1.8) and boundary conditions (1.2)–(1.6), which is evidence of the correctness of the formulated boundary-value problems and the effectiveness of the asymptotic method of solution.

3. Solution of the formulated boundary-value problems

Satisfying the contact conditions for the layers (1.8), we determine the 6(n – 1) integration functions

$$\zeta_i = \varphi_i/h : \sigma_{zz0}^{(i,s)} = \Big[F_{xx}^{(i+1,s)}\left((D_{24}^{(i)}+1)\psi_{ix} - D_{25}^{(i)}\psi_{iy}\right) + F_{yy}^{(i+1,s)}\left((D_{15}^{(i)}+1)\psi_{iy} - D_{14}^{(i)}\psi_{ix}\right) +$$
$$+F_{zz}^{(i+1,s)}\left(D_{14}^{(i)}D_{25}^{(i)} - (D_{15}^{(i)}+1)(D_{24}^{(i)}+1)\right)\Big]\Big/\Delta_D^{(i)}$$
$$\sigma_{yz0}^{(i,s)} = \Big[F_{xx}^{(i+1,s)}\left(D_{25}^{(i)} - D_{23}^{(i)}\psi_{ix}\right) + F_{yy}^{(i+1,s)}\left(D_{13}^{(i)}\psi_{ix} - (D_{15}^{(i)}+1)\right) +$$
$$+F_{zz}^{(i+1,s)}\left((D_{15}^{(i)}+1)D_{23}^{(i)} - D_{13}^{(i)}D_{25}^{(i)}\right)\Big]\Big/\Delta_D^{(i)}$$
$$\sigma_{xz0}^{(i,s)} = \Big[F_{xx}^{(i+1,s)}\left(D_{23}^{(i)}\psi_{iy} - (D_{24}^{(i)}+1)\right) + F_{yy}^{(i+1,s)}\left(D_{14}^{(i)} - D_{13}^{(i)}\psi_{iy}\right) +$$
$$+F_{zz}^{(i+1,s)}\left(D_{13}^{(i)}(D_{24}^{(i)}+1) - D_{14}^{(i)}D_{23}^{(i)}\right)\Big]\Big/\Delta_D^{(i)}$$
$$u_0^{(i+1,s)} = u_0^{(1,s)} + \sum_{k=1}^{i}\zeta_k\left(\tilde\tau_5^{(k,s)} - \tilde\tau_5^{(k+1,s)}\right) + \sum_{k=1}^{i}\left(u_*^{(k,s)}(\zeta_k) - u_*^{(k+1,s)}(\zeta_k)\right)$$
$$(u,\upsilon,w;5,4,3), \quad i = 1,2,\ldots,n-1 \tag{3.1}$$

Here,

$$D_{1k}^{(i)} = A_{1k}^{(i)}\psi_{ix} + A_{6k}^{(i)}\psi_{iy}, \quad D_{1k}^{*(i)} = A_{1k}^{(i+1)}\psi_{ix} + A_{6k}^{(i+1)}\psi_{iy} \quad (1,2;x,y)$$
$$F_{xx}^{(i+1,s)} = D_{13}^{*(i)}\sigma_{zz0}^{(i+1,s)} + D_{14}^{*(i)}\sigma_{yz0}^{(i+1,s)} + (D_{15}^{*(i)}+1)\sigma_{xz0}^{(i+1,s)} + \tau_{x*}^{(i,s)}$$
$$F_{yy}^{(i+1,s)} = D_{23}^{*(i)}\sigma_{zz0}^{(i+1,s)} + (D_{24}^{*(i)}+1)\sigma_{yz0}^{(i+1,s)} + D_{25}^{*(i)}\sigma_{xz0}^{(i+1,s)} + \tau_{y*}^{(i,s)}$$
$$F_{zz}^{(i+1,s)} = \sigma_{zz0}^{(i+1,s)} + \sigma_{yz0}^{(i+1,s)}\psi_{iy} + \sigma_{xz0}^{(i+1,s)}\psi_{ix} + \tau_{z*}^{(i,s)}$$
$$\tau_{j*}^{(i,s)} = \left(\sigma_{jx*}^{(i+1,s)}(\zeta_i) - \sigma_{jx*}^{(i,s)}(\zeta_i)\right)\psi_{ix} + \left(\sigma_{iy*}^{(i+1,s)}(\zeta_i) - \sigma_{jy*}^{(i,s)}(\zeta_i)\right)\psi_{iy} +$$
$$+\left(\sigma_{jz*}^{(i+1,s)}(\zeta_i) - \sigma_{jz*}^{(i,s)}(\zeta_i)\right), \quad j = x,y,z$$
$$\Delta_D^{(i)} = D_{14}^{(i)}D_{25}^{(i)} - (D_{15}^{(i)}+1)(D_{24}^{(i)}+1) + \psi_{iy}\left(D_{23}^{(i)}(D_{15}^{(i)}+1) - D_{13}^{(i)}D_{25}^{(i)}\right) +$$
$$+\psi_{ix}\left(D_{13}^{(i)}(D_{24}^{(i)}+1) - D_{14}^{(i)}D_{23}^{(i)}\right), \quad i = 1,2,\ldots,n-1 \tag{3.2}$$

The three integration functions $u_0^{1,s}(\xi,\eta)$, $\vartheta_0^{1,s}(\xi,\eta)$, $w_0^{1,s}(\xi,\eta)$ are determined from boundary conditions (1.2) specified on the face $z=\varphi_0\,(x,y)$ of the first ($i=1$) layer of the body

$$\zeta_0=\varphi_0/h:\ u_0^{(1,s)}=u_x^{-(s)}-\zeta_0\tilde{\tau}_5^{(1,s)}-u_*^{(1,s)}\qquad (u,\upsilon,w;5,4,3;u_x^-,u_y^-,u_z^-)$$
$$u_x^{-(0)}=u^-\Big/l,\quad u_x^{-(s)}=0,\quad s\neq 0\quad \left(x,y,z;u^-,\upsilon^-,w^-\right) \tag{3.3}$$

The remaining three integration functions $\sigma_{jz0}^{(n,s)}(\xi,\eta)(j=x,y,z)$ are determined by one of the groups of boundary conditions (1.3), (1.4), (1.5), (1.6) specified on the face $z=\varphi_n\,(x,y)$ of the last layer ($i=n$) of the body.

1°. If the conditions of the first boundary-value problem of the theory of elasticity (1.3) are specified on the face $z=\varphi_n\,(x,y)$, then the integration functions $\sigma_{jz0}^{(n,s)}(\xi,\eta)j=x,y,z$ are determined using formulae (3.1) and (3.2), where it is necessary to take $i=n$ and

$$F_{jx}^{(n+1,s)}=\lambda_n\Phi_{\vartheta j}^{(s)}-\sigma_{jx*}^{(n,s)}(\zeta_n)\psi_{nx}-\sigma_{jy*}^{(n,s)}(\zeta_n)\psi_{ny}-\sigma_{jz*}^{(n,s)}(\zeta_n)$$
$$\Phi_{\vartheta j}^{(0)}=\varepsilon\Phi_{\vartheta j},\quad \Phi_{\vartheta j}^{(s)}=0\ \ s>0;\quad j=x,y,z \tag{3.4}$$

2°. If kinematic conditions (1.4) are specified on the face, it is more convenient to represent the solution of the boundary-value problem in the matrix form

$$\mathrm{col}\left[\sigma_{zz0}^{(1,s)},\sigma_{yz0}^{(1,s)},\sigma_{xz0}^{(1,s)},u_0^{(1,s)},\upsilon_0^{(1,s)},w_0^{(1,s)},\ldots,\sigma_{zz0}^{(n,s)},\sigma_{yz0}^{(n,s)},\sigma_{xz0}^{(n,s)},u_0^{(n,s)},\upsilon_0^{(n,s)},w_0^{(n,s)}\right]=$$
$$=\|c_{mk}\|_{(6n)\times(6n)}^{-1}\times\|b_{k1}\|_{(6n)\times 1} \tag{3.5}$$

The elements of the square matrix, resulting from the contact conditions between the layers (1.8), have the form

$$c_{mm}=D_{13}^{(i)},\quad c_{m(m+1)}=D_{14}^{(i)},\quad c_{m(m+2)}=D_{15}^{(i)}+1$$
$$c_{m(m+6)}=-D_{13}^{*(i)},\quad c_{m(m+7)}=-D_{14}^{*(i)},\quad c_{m(m+8)}=-D_{15}^{*(i)}-1$$
$$c_{(m+1)m}=D_{23}^{(i)},\quad c_{(m+1)(m+1)}=D_{23}^{(i)}+1,\quad c_{(m+1)(m+2)}=D_{25}^{(i)}$$
$$c_{(m+1)(m+6)}=-D_{23}^{*(i)},\quad c_{(m+1)(m+7)}=-D_{24}^{*(i)}-1,\quad c_{(m+1)(m+8)}=-D_{25}^{*(i)}$$
$$c_{(m+2)m}=-c_{(m+2)(m+6)}=1,\quad c_{(m+2)(m+1)}=-c_{(m+2)(m+7)}=\psi_{iy}$$
$$c_{(m+2)(m+2)}=-c_{(m+2)(m+8)}=\psi_{ix},\quad c_{(m+3)m}=\zeta_iA_{53}^{(i)}$$
$$c_{(m+3)(m+1)}=\zeta_iA_{54}^{(i)},\quad c_{(m+3)(m+2)}=\zeta_iA_{55}^{(i)},\quad c_{(m+3)(m+3)}=-c_{(m+3)(m+9)}=1$$
$$c_{(m+3)(m+6)}=-\zeta_iA_{53}^{(i+1)},\quad c_{(m+3)(m+7)}=-\zeta_iA_{54}^{(i+1)},\quad c_{(m+3)(m+8)}=-\zeta_iA_{55}^{(i+1)}$$
$$c_{(m+4)m}=\zeta_iA_{43}^{(i)},\quad c_{(m+4)(m+1)}=\zeta_iA_{44}^{(i)},\quad c_{(m+4)(m+2)}=\zeta_iA_{45}^{(i)}$$
$$c_{(m+4)(m+4)}=-c_{(m+4)(m+10)}=1,\quad c_{(m+4)(m+6)}=-\zeta_iA_{43}^{(i+1)},\quad c_{(m+4)(m+7)}=-\zeta_iA_{44}^{(i+1)}$$
$$c_{(m+4)(m+8)}=-\zeta_iA_{45}^{(i+1)},\quad c_{(m+5)m}=\zeta_iA_{33}^{(i)},\quad c_{(m+5)(m+1)}=\zeta_iA_{34}^{(i)},\quad c_{(m+5)(m+2)}=\zeta_iA_{35}^{(i)}$$
$$c_{(m+5)(m+5)}=-c_{(m+5)(m+11)}=1,\quad c_{(m+5)(m+6)}=-\zeta_iA_{33}^{(i+1)},\quad c_{(m+5)(m+7)}=-\zeta_iA_{34}^{(i+1)}$$
$$c_{(m+5)(m+8)}=-\zeta_iA_{35}^{(i+1)},\quad c_{m(m+k)}=c_{(m+1)(m+k)}=c_{(m+2)(m+k)}=0,\quad k=3,4,5,\quad k>8$$
$$c_{(m+3)(m+k)}=0,\quad k=4,5,\quad k>10;\quad c_{(m+4)(m+k)}=0,\quad k=3,5,9,\quad k>10$$
$$c_{(m+5)(m+k)}=0,\quad k=3,4,9,10;\quad k>11 \tag{3.6}$$

Henceforth,

$$m=6i-5,\quad i=1,2,\ldots,n-1 \tag{3.7}$$

The elements, stipulated by boundary conditions (1.2) and (1.4) will be

$$c_{(6n-p)q}=\zeta_0A_{p(q+2)}^{(1)},\quad p=3,4,5;\quad q=1,2,3$$
$$c_{(6n-p)(6n-5)}=\zeta_nA_{(p+3)(8-q)}^{(n)},\quad p=0,1,2;\quad q=3,4,5$$
$$c_{(6n-5)4}=c_{(6n-4)5}=c_{(6n-3)6}=c_{(6n-2)(6n-2)}=c_{(6n-1)(6n-1)}=c_{(6n)(6n)}=1$$
$$c_{(6n-5)k}=0,\quad k>4;\quad c_{(6n-4)k}=0,\quad k=4,\quad k>5;\quad c_{(6n-3)k}=0,\quad k=4,5,k>6$$
$$\zeta_0=\varphi_0/h,\quad \zeta_n=\varphi_n/h \tag{3.8}$$

and the elements of the column matrix are

$$\begin{aligned}
&b_m = \tau_{x*}^{(i,s)}, \quad b_{m+1} = \tau_{y*}^{(i,s)}, \quad b_{m+2} = \tau_{z*}^{(i,s)} \\
&b_{m+3} = u_*^{(i+1,s)}(\zeta_i) - u_*^{(i,s)}(\zeta_i), \quad b_{m+4} = \upsilon_*^{(i+1,s)}(\zeta_i) - \upsilon_*^{(i,s)}(\zeta_i) \\
&b_{m+5} = w_*^{(i+1,s)}(\zeta_i) - w_*^{(i,s)}(\zeta_i), \quad b_{6n-5} = u_*^{-(s)} - u_*^{(1,s)}(\zeta_0) \\
&b_{6n-4} = \upsilon^{-(s)} - \upsilon_*^{(1,s)}(\zeta_0), \quad b_{6n-3} = w^{-(s)} - w_*^{(1,s)}(\zeta_0), \quad b_{6n-2} = u_*^{+(s)} - u_*^{(n,s)}(\zeta_n) \\
&b_{6n-1} = \upsilon^{+(s)} - \upsilon_*^{(n,s)}(\zeta_n), \quad b_{(6n)} = w^{+(s)} - w_*^{(n,s)}(\zeta_n) \\
&\tau_j^{(i,s)} = (\sigma_{jx*}^{(i+1,s)}(\zeta_i) - \sigma_{jx*}^{(i,s)}(\zeta_i))\psi_{ix} + (\sigma_{iy*}^{(i+1,s)}(\zeta_i) - \sigma_{jy*}^{(i,s)}(\zeta_i))\psi_{iy} \\
&+ (\sigma_{jz*}^{(i+1,s)}(\zeta_i) - \sigma_{jz*}^{(i,s)}(\zeta_i)), \quad j = x,\ y,\ z \\
&u^{\pm(0)} = u_x^{\pm}/l \quad u^{\pm(s)} = 0 \quad s \neq 0 \quad (x, y, z; u, \upsilon, w)
\end{aligned} \tag{3.9}$$

3°. If one of the combinations of the mixed conditions (1.5) or (1.6) is specified on the face of a thin body, the matrix elements respectively change in the last rows in the case of conditions (1.5)

$$\begin{aligned}
&c_{(6n)(6n-5)} = 1, \quad c_{(6n)(6n-4)} = \psi_{ny}, \quad c_{(6n)(6n-3)} = \psi_{nx}, \quad b_{6n} = \overline{\tau}_z^{(n,s)} \\
&\overline{\tau}_j^{(n,s)} = \lambda_n \Phi_{\vartheta j}^{(s)} - \sigma_{jx*}^{(n,s)}(\zeta_n)\psi_{nx} - \sigma_{jy*}^{(n,s)}(\zeta_n)\psi_{ny} - \sigma_{jz*}^{(n,s)}(\zeta_n) \\
&\Phi_{\vartheta j}^{(0)} = \varepsilon\Phi_{\vartheta j}, \quad \Phi_{\vartheta j}^{(s)} = 0; \quad s \neq 0, \quad j = x, y, z
\end{aligned} \tag{3.10}$$

and, in the two penultimate rows, in the case of conditions (1.6)

$$\begin{aligned}
&c_{(6n-2)(6n-5)} = D_{13}^{(n)}, \quad c_{(6n-2)(6n-4)} = D_{14}^{(n)}, \quad c_{(6n-2)(6n-3)} = D_{15}^{(n)} + 1 \\
&c_{(6n-1)(6n-5)} = D_{23}^{(n)}, \quad c_{(6n-1)(6n-4)} = D_{24}^{(n)} + 1, \quad c_{(6n-1)(6n-3)} = D_{25}^{(n)} \\
&b_{6n-2} = \overline{\tau}_x^{(n,s)}, \quad b_{6n-1} = \overline{\tau}_y^{(n,s)}, \quad c_{(6n-2)k} = c_{(6n-1)k} = 0, \quad k < 6n-5
\end{aligned} \tag{3.11}$$

Hence the solutions of the above boundary-value problems are represented by the recursion formulae (2.5)–(2.8), where the unknown integration functions for boundary conditions (1.2) and (1.3) are given by formulae (3.1)–(3.4) and, for the boundary conditions (1.2), (1.4), (1.5), (1.6) using the matrix formula (3.5) with the corresponding elements (3.6)–(3.11). Note that the solution of problem (1.2), (1.3) can also be obtained using matrix formula (3.5) if the elements of the last three rows are chosen using formulae (3.10) and (3.11).

These solutions enable us to calculate the components of the stress tensor and the displacement vector with any asymptotic accuracy $O(\varepsilon^S)$ if all the given functions, mam the equations of the surfaces of the layers $\varphi_0\,(x, y)$,. . .,$\varphi_n\,(x, y)$, the physicomechanical coefficients a_{ij} (x, y) and $\alpha_{m,k}\,(x, y)$ and the functions $\Phi_{\upsilon j}$, $u_j^{\pm}(j = x, y, z)$, defined on the surfaces, are continuous and bounded and have continuous and bounded partial and mixed derivatives of the order S. At the same time, their variability must be of an order not greater than unity[14]. If $\Phi_{\upsilon j}$, $u_j^{\pm}(j = x, y, z)$ is a polynomial in x and y and all the other given functions are constant, then, after a finite number of iteration steps, the process terminates and a mathematically exact solution in polynomials in x, y and z is obtained.

Note that the necessary condition for the existence and uniqueness of the solutions of the boundary-value problems formulated in Section 1 has the form

$$\det\|c_{mk}\|_{(6n)\times(6n)} \neq 0 \tag{3.12}$$

which, in the case of existing materials, is always satisfied on the basis of Hooke's elasticity relations.

The recursion formulae derived are simultaneously a ready-made algorithm for a universal computer program which enables one to obtain the analytical solutions of the above boundary-value problems with any specified accuracy for thin bodies such as laminated plates and shells of variable thickness.

4°. It is well known that strong earthquakes are associated with the tectonic shifts of the lithospheric plates of the Earth's crust and they are localized in the zone where the corresponding (Eurasian, African, Arabian, Pacific, North American, etc.) plates come into contact (collide). The longitudinal dimensions of the plates vary from hundreds to several thousand kilometres while their thickness is much smaller.[16] In the last decade, displacements of the points of the surface of the lithospheric plates have been measured using satellite GPS (global positioning system's) and, on the basis of these measurements, an attempt has been made to evaluate the accumulated strain energy in the collision zone. The three-dimensional problem in the theory of elasticity which arises is non-classical in the sense that boundary conditions are only specified on one face and they are supersaturated, that is, the stresses are equal to zero, and the values of the displacements of points of the surface are simultaneously given (GPS data, seismology stations)[17,18]. The general integral (2.7), (2.8) and the matrix representation (3.5) of the solutions of boundary-value problems enable one to solve such "Hadamard ill-posed" problems. Of all the combinations of boundary conditions (1.2)–(1.6), conditions (1.3) and (1.4), which are set on the one and the same surface simultaneously, correspond to this situation. Satisfying these conditions, for the matrix elements (3.5) we obtain the values (3.6)–(3.8) where it is only necessary to change the elements of the sixth, fifth and fourth rows from the end. These elements take the following values

$$
\begin{aligned}
&c_{(6n-5)(6n-5)} = 1, \quad c_{(6n-5)(6n-4)} = \psi_{ny}, \quad c_{(6n-5)(6n-3)} = \psi_{nx} \\
&c_{(6n-3)(6n-5)} = D_{13}^{(n)}, \quad c_{(6n-3)(6n-4)} = D_{14}^{(n)}, \quad c_{(6n-3)(6n-3)} = D_{15}^{(n)} + 1 \\
&c_{(6n-4)(6n-5)} = D_{23}^{(n)}, \quad c_{(6n-4)(6n-4)} = D_{24}^{(n)} + 1, \quad c_{(6n-4)(6n-3)} = D_{25}^{(n)} \\
&b_{6n-5} = \overline{\tau}_z^{(n,s)}, \quad b_{6n-4} = \overline{\tau}_y^{(n,s)}, \quad b_{6n-3} = \overline{\tau}_x^{(n,s)}, \quad c_{(6n-j)k} = 0, \quad j = 3,4,5,\ k < 6n-5
\end{aligned}
\tag{3.13}
$$

As a result, the solution of the problem is determined by the recursion formulae (2.5)–(2.8), (3.6)–(3.8) taking account of expressions (3.13). The solutions of the boundary-value problems have a fairly wide spectrum of applications. For example, knowing the structure of the lithospheric plates at a given locality and the GPS data, we can establish the character of the stress-strain state at a certain depth which is extremely important in seismology and in the prediction of earthquakes.

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